COVER PAGE

Numerical Analysis Qualifying Examination

Friday, January 4, 2019

9:30am – 11:30am

C-304 Wells Hall

Your Sign-Up Number: ________________________

Note: Attach this cover page to the paperwork you are submitting to be graded. **This number should be the only identification appearing on all of your paperwork – DO NOT WRITE YOUR NAME on any of the paperwork you are submitting.**

1. (10 points) Let $A, B \in C^{m\times m}$ be arbitrary matrices. Show that

$$\|AB\|_F \leq \|A\|_2 \|B\|_F,$$

where $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the 2-norm and Frobenius norm, respectively.

2. (15 points) Fix $0 < \varepsilon < 1$ and suppose that $A \in R^{m \times m}$ is symmetric and nonsingular. Show that if $\|A - I\|_F \geq \varepsilon$, then $\|A^{-1} - I\|_F \geq \frac{\varepsilon}{2}$, where $\|\cdot\|_F$ denotes the Frobenius norm.

3. (15 points) Let $\varepsilon > 0$ be given, $k << \min(m, n)$, $A \in R^{m\times n}$, $C \in R^{m\times k}$, and $B \in R^{k\times n}$. Assume that

$$\|A - CB\| \leq \epsilon,$$

where $\|\cdot\|$ denotes the matrix 2-norm, and B and C have rank k. Further suppose that A is not available, and only B and C are available. **Without forming** the product of C and B, design an efficient algorithm to compute an approximate reduced QR of A so that the following holds,

$$\|A - QR\| \leq \varepsilon,$$

where Q is an orthonormal matrix and R is upper triangular.

4. (10 points) Show that if $A \in \mathcal{R}^{n\times n}$ is symmetric, then for $k = 1$ to n,

$$\lambda_k(A) = \max_{\dim(S)=k} \min_{\mathbf{0}\neq\mathbf{y}\in S} \frac{\mathbf{y}^T A\mathbf{y}}{\mathbf{y}^T\mathbf{y}},$$

where S is a subspace of $\mathcal{R}^n$, and $\lambda_k(A)$ designates the kth largest eigenvalue of A so that these eigenvalues are ordered,

$$\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A).$$

5. Let $A \in \mathcal{R}^{m \times n}$, $\text{rank}(A) = r$, and $\mathbf{b} \in \mathcal{R}^m$, and consider the system $A\mathbf{x} = \mathbf{b}$ with unknown $\mathbf{x} \in \mathcal{R}^n$. Making no assumption about the relative sizes of n and m, we formulate the following least-squares problem:

   of all the $\boldsymbol{x} \in \mathcal{R}^n$ that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$, find the one for which $\|\boldsymbol{x}\|_2$ is minimized.

   (a) (5 points) Show that the set Γ of all minimizers of the least-squares function is a closed convex set:

   $$\Gamma = \{\mathbf{x} \in \mathcal{R}^n : \|A\mathbf{x} - \mathbf{b}\|_2 = \min_{\mathbf{v} \in \mathcal{R}^n} \|A\mathbf{v} - \mathbf{b}\|_2\}.$$

   (b) (5 points) Show that the minimum-norm element in Γ is unique.

   (c) (10 points) Show that the minimum norm solution is $\mathbf{x} = A^+\mathbf{b} = V\Sigma^+U^*\mathbf{b}$, where $A = U\Sigma V^*$, and Σ^+ is the pseudo-inverse of Σ.

6. Let $A \in R^{n\times n}$ be symmetric and positive definite and $n = j + k$. Partition A into the following 2 by 2 blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is $j \times j$ and A_{22} is $k \times k$. Let R_{11} be the Cholesky factor of A_{11}: $A_{11} = R_{11}^T R_{11}$, where R_{11} is upper triangular with positive main-diagonal entries. Let $R_{12} = \left(R_{11}^{-1}\right)^T A_{12}$ and let $\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$.

(a) (5 points) Prove that A_{11} is positive definite.

(b) (5 points) Prove that

$$\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}.$$

(c) (5 points) Prove that $\tilde{A}_{22}$ is positive definite.

7. Consider the following integration formula,

$$u(x) = \int_0^1 G(x,y)f(y)dy, \tag{1}$$

where $f \in C[0,1]$ and $G(x,y)$ is given by

$$G(x,y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x \\ x(1-y) & \text{if } x \le y \le 1 \end{cases} \tag{2}$$

Partition [0,1] into $n+1$ equal subintervals with mesh size $h = \frac{1}{n+1}$: $x_j = j * h$, $\hat{u}_j \approx u_j = u(x_j)$ for $0 \le j \le n+1$. We also introduce the following vector notation $U = (u_0, u_1, u_2, \cdots, u_n, u_n)^t$, and $F = (f_0, f_1, f_2, \cdots, f_n)^t$, and $\hat{U} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_n)^t$.

(a) (5 points) To evaluate the vector $\hat{U}$, we may approximate this integral formula (1) by the Riemann sum based on the above uniform partition,

$$\hat{u}_i = \sum_{j=0}^{n} G(x_i, y_j) f(y_j) h,$$

which will lead to a matrix-vector product $\hat{U} = \hat{G}F$ in terms of a matrix $\hat{G}$ defined by

$$\hat{G} = (h * G(x_i, y_j))_{0 \le i \le n, 0 \le j \le n}$$

and the vector F. Write down this matrix-vector product to obtain the vector $\hat{U}$ from the Riemann sum. Show that the complexity of this matrix-vector product is $O(n^2)$.

(b) (10 points) Based on the above uniform partition, use the structure of the Green's function G to design an $O(n)$ algorithm to compute the vector $\hat{U}$.